Composite Hugoniot Synthesis Using the Theory of Mixtures

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ABSTRACT

The contemporary theory of mixtures is employed to predict the response of composites to shock loading. The composite may consist of any finite number of constituents. Jump equations expressing balance of mass, momentum, and energy for each of the constituents are presented, following the approach of Kelly [1]. These equations are then specialized to apply to a uniaxial shock wave running into undisturbed material. The resulting theory is shown to encompass constituents which undergo phase transformations and exchange thermal energy within the shock surface.

INTRODUCTION

RECENTLY, Tsou and Chou have presented a so-called control-volume approach to Hugoniot synthesis for composite materials (References [2] and [3]). The usual approach to this problem (cf. Lysne [4]; Anderson, *et. al.*, [5]; Torvik, [6]) evolves around a mass-weighted average of the constituent Hugoniots to obtain the composite Hugoniot. The approach of Tsou and Chou is somewhat different in that they employ the jump relations for mass, momentum, and energy for the composite together with certain mixture concepts relating constituent and composite densities, energies, etc. to obtain the composite Hugoniot. Tsou and Chou have successfully compared their theory with experimental results for two alloys: copper-zinc and beryllium-aluminum. Their theory is somewhat restricted, however. The possibilities of chemical reaction among the constituents, or phase transformations of a constituent are not considered. Also, it must be assumed that either (1) the constituents come into thermal equilibrium with each other within the shock surface, or (2) no heat is transferred between constituents at any time. Neither

of these assumptions seems too likely, and indeed the true thermal condition probably lies somewhere between these two limiting states.

In this paper we present equations to represent the composite as a classical mixture. We shall show that the theory of Tsou and Chou can be obtained as a special case of the more general mixture theory, and that the mixture approach will yield the necessary tools to consider phase transformations, chemical reactions, and partial heat transfer among the constituents within the shock surface. The mixture theory approach has previously been employed by Riney, *et. al.* [7] and Davis [8] to represent geologic materials subject to shock loading. A large volume of work concerning applications of mixture theory to other types of loading for various materials has also appeared (e.g., Knops and Steel [9]).

BASIC EQUATIONS

Mixture theory is founded on a set of concepts elucidated by Truesdell and Toupin [10]. Each constituent of the mixture is considered a separate continuum which may be acted upon by exterior actions and by other constituents. It is postulated that the mean motion of the whole mixture is a mathematical consequence of the motions of the individual constituents and, moreover, that this mean motion must obey the laws which govern the motion of a single material. Thus, the mixture as a whole does not know whether it is composed of only one or of several constituents. Since each constituent is considered a continuum, every place in the mixture may be simultaneously occupied by particles of any or all of the various constituents. Whereas mass, momentum, and energy of each constituent are not necessarily conserved, the mixture mass, momentum, and energy are.

We shall designate the mixture constituents by S_α, $\alpha = 1, 2, \ldots, k$. The total number of constituents, k, is left arbitrary. By ρ_α we denote the mass density of S_α. Since S_α is a continuum, ρ_α is a bulk density; that is, the mass of S_α per unit volume of the mixture. The mixture density, designated ρ, is given by

$$\rho = \sum_\alpha \rho_\alpha \tag{1}$$

Henceforth, quantities lacking Greek subscripts will refer to the mean properties of the mixture, and the symbol $\sum_\alpha$ will always indicate the summation over $\alpha = 1, 2, \ldots, k$.

The particle velocity of the constituent S_α is given by

$$\mathbf{v}_\alpha = \left. \frac{\partial \mathbf{x}_\alpha}{\partial t} \right|_{X_\alpha = \text{constant}} \tag{2}$$

Here $\mathbf{x}_\alpha$ is the place occupied by the particle X_α at time t. We also introduce the concentration or mass fraction, c_α, for the constituent S_α.

$$c_\alpha = \rho_\alpha/\rho \tag{3}$$

Then the mean velocity of the mixture, $\mathbf{v}$, is given by the mass-weighted average of the individual velocities.

$$\mathbf{v} = \sum_\alpha c_\alpha \mathbf{v}_\alpha \tag{4}$$

We also define the diffusion velocity, $\mathbf{u}_\alpha$, of the constituents S_α as

$$\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v} \tag{5}$$

Multiplying Equation (5) by ρ_α and summing over α leads to the identity

$$\sum_\alpha \rho_\alpha \mathbf{u}_\alpha \equiv 0 \tag{6}$$

We next define a partial stress tensor, $\mathbf{T}_\alpha$, for the constituent S_α, as well as a specific internal energy, ϵ_α, and a heat flux vector, $\mathbf{h}_\alpha$. The total stress, $\mathbf{T}$, internal energy, ϵ, and heat flux, $\mathbf{h}$, for the mixture are then given by the following relations.

$$\mathbf{T} = \sum_\alpha (\mathbf{T}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) \tag{7}$$

$$\epsilon = \sum_\alpha c_\alpha \left(\epsilon_\alpha + \frac{1}{2} u_\alpha^2\right) \tag{8}$$

$$\mathbf{h} = \sum_\alpha \left\{\mathbf{h}_\alpha + \mathbf{T}_\alpha \mathbf{u}_\alpha - \rho_\alpha \left(\epsilon_\alpha + \frac{1}{2} u_\alpha^2\right) \mathbf{u}_\alpha\right\} \tag{9}$$

In the above equations, $\otimes$ denotes the tensor product of two vectors, while u_α^2 denotes the inner product of $\mathbf{u}_\alpha$ with itself. Motivation for Equations (7), (8), and (9) may be found in Reference [10].

We shall now set down three equations which govern the balance of mass, momentum, and energy for each of the mixture constituents, S_α, crossing a shock surface. These relations were first derived by Kelly in Reference [1]. We denote by U the spatial velocity of the shock, assumed normal to the shock surface. Letting Ψ represent any quantity which approaches continuous limits on both sides of the shock surface, we denote the jump in Ψ by

$$[\Psi] = \Psi^+ - \Psi^-$$

where Ψ^+ represents the limiting value of Ψ immediately in front of the shock and Ψ^- the value immediately behind.

The balance of mass jump relation for the constituent S_α crossing the

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shock surface can then be written as

$$[\rho_\alpha(\mathbf{U} - \mathbf{v}_\alpha) \cdot \mathbf{N}] + \underset{\sigma}{\hat{c}_\alpha} = 0 \tag{10}$$

In this equation, $\mathbf{N}$ denotes the unit normal to the shock surface and $\underset{\sigma}{\hat{c}_\alpha}$ is the surface mass supply for S_α on the shock surface. Except for the term $\underset{\sigma}{\hat{c}_\alpha}$, Equation (10) is identical to the usual mass jump equation for a single material. The extra term allows for mass transfer among the mixture constituents. It accounts for the growth (or loss) of S_α mass within the shock surface.

The balance of momentum jump equation for S_α is given by

$$[\rho_\alpha \mathbf{v}_\alpha \otimes (\mathbf{U} - \mathbf{v}_\alpha) + \mathbf{T}_\alpha] \cdot \mathbf{N} + \underset{\sigma}{\hat{\mathbf{m}}_\alpha} = 0 \tag{11}$$

Here $\underset{\sigma}{\hat{\mathbf{m}}_\alpha}$ is called the surface momentum supply for S_α. It accounts for the growth or decay of S_α momentum within the shock surface. Finally, the energy jump equation for the constituent S_α is given by

$$\left[\rho_\alpha\left(\epsilon_\alpha + \frac{1}{2}v_\alpha^2\right)\left(\mathbf{U} - \mathbf{v}_\alpha\right) + \mathbf{T}_\alpha \mathbf{v}_\alpha + \mathbf{h}_\alpha\right] \cdot \mathbf{N} + \underset{\sigma}{\hat{\epsilon}_\alpha} = 0 \tag{12}$$

where $\underset{\sigma}{\hat{\epsilon}_\alpha}$ is the surface energy supply for S_α within the shock surface. Like the mass and momentum supplies previously defined, $\underset{\sigma}{\hat{\epsilon}_\alpha}$ allows for energy exchange among the mixture constituents within the shock.

Equations (10), (11), and (12) are correct balance statements crossing any surface within the mixture, shock or not. If we are to use them effectively below, however, we must assume the existance of a stable shock front passing through the mixture. This being done, we point out that the jump equations apply only to the limit values of the various quantities ρ_α, $\mathbf{v}_\alpha$, etc. leading and following the shock. No assumption of a perfect step pulse is made, nor is it considered appropriate for many mixtures.

Following the usual program of mixture theory, we sum Equations (10), (11), and (12) over the range $\alpha = 1, 2, \ldots, k$ to obtain the mixture jump equations. These equations express conservation of mass, momentum, and energy in the whole mixture. We employ Equations (1) and (3) through (9) as well as the postulate that the mean motion of the mixture must obey the jump conditions for a single material. The mass, momentum, and energy equations for the mixture become respectively

$$\sum_\alpha \underset{\sigma}{\hat{c}_\alpha} \equiv 0 \tag{13}$$

$$\sum_\alpha \underset{\sigma}{\hat{\mathbf{m}}_\alpha} \equiv \mathbf{0} \tag{14}$$

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$$\sum_\alpha \hat{\epsilon}_\alpha \equiv 0 \qquad (15)$$

For a more detailed derivation of these three relations, see Reference [1].

Next we wish to make a distinction between the physical configuration of the composite material and the idealized mixture configuration outlined above. The mixture equations are based on the concept that each constituent is itself a continuum, and any place in the mixture may be simultaneously occupied by all of the constituents. It is unlikely that this will be true of the physical configuration of the composite, where generally the constituents will remain distinct from one another. The two configurations, physical and mixture, should not be confused. Rather we desire to employ both in order to best represent the composite material. To accomplish this we introduce the volume fraction n_α for the constituent S_α. We define n_α as the ratio of the volume of S_α in the physical configuration per unit volume of the mixture. Clearly then

$$\sum_\alpha n_\alpha \equiv 1 \qquad (16)$$

We view n_α on a macroscopic scale only, determining it on an appropriately sized neighborhood of the point in question.

We can now define certain quantities in the physical configuration. The crystal density, $\tilde{\rho}_\alpha$, for the constituent α is defined by

$$\rho_\alpha = n_\alpha \tilde{\rho}_\alpha \qquad (17)$$

Clearly, $\tilde{\rho}_\alpha$ represents the mass of S_α per unit volume of that constituent in the physical configuration. We assume $\tilde{\rho}_\alpha$ remains finite so that ρ_α will vanish whenever n_α does. Next, we assume the constituents are isotropically arranged in the physical configuration, so that surfaces intersecting the composite will consist of constituent S_α over n_α of their area. Then we can define a crystal stress, $\widetilde{\mathbf{T}}_\alpha$, for S_α given by

$$\mathbf{T}_\alpha = n_\alpha \widetilde{\mathbf{T}}_\alpha \qquad (18)$$

Similarly, we have a crystal heat flux, $\tilde{\mathbf{h}}_\alpha$.

$$\mathbf{h}_\alpha = n_\alpha \tilde{\mathbf{h}}_\alpha \qquad (19)$$

We may also introduce a crystal internal energy, $\tilde{\epsilon}_\alpha$. Since the internal energy is measured per unit mass of S_α rather than per unit volume, we see that ϵ_α and $\tilde{\epsilon}_\alpha$ will be identical, provided surface energy in the physical configuration is ignored. Finally, the mixture density, ρ, can be related to the constituent crystal densities, $\tilde{\rho}_\alpha$, by

$$\frac{1}{\rho} = \sum_\alpha \frac{c_\alpha}{\tilde{\rho}_\alpha} \qquad (20)$$

as is easily proved by Equations (3) and (17).

ASSUMPTIONS

In this section we shall make several specializing assumptions which will lead to certain particular cases of the general mixture theory described above. We shall first assume that the stress in all constituents is sufficiently high so that a hydrodynamic stress state prevails. That is

$$\mathbf{T}_\alpha = -P_\alpha \mathbf{1} \tag{21}$$

where P_α is called the partial pressure for the constituent S_α. We may also define a crystal pressure, $\widetilde{P}_\alpha$, given by

$$\widetilde{\mathbf{T}}_\alpha = -\widetilde{P}_\alpha \mathbf{1} \tag{22}$$

Then from Equation (18) we have

$$P_\alpha = n_\alpha \widetilde{P}_\alpha \tag{23}$$

Next, we restrict further consideration to plane shock waves running into undisturbed material. The constituent density leading the shock, ρ_α^+, will thus be the initial density, ρ_{α_0}. Also we need only consider the components of velocity and heat flux normal to the planar shock surface. These will be designated v_α and h_α. Further, we let the velocity, internal energy, pressure, and heat flux leading the shock vanish. That is

$$v_\alpha^+ = \epsilon_\alpha^+ = P_\alpha^+ = h_\alpha^+ = 0, \qquad \alpha = 1, 2, \ldots, k \tag{24}$$

These assumptions are made solely for the sake of convenience. The resulting equations are somewhat more cumbersome if the above assumptions are not made, but they offer no more real difficulty in derivation.

Using the above assumptions, the jump equations [Equations (10), (11), and (12)] can be written as

$$\rho_\alpha^-(U - v_\alpha^-) = \rho_{\alpha_0} U + \underset{\sigma}{\hat{c}}_\alpha \tag{25}$$

$$P_\alpha^- + \underset{\sigma}{\hat{m}}_\alpha = \rho_\alpha^-(U - v_\alpha^-) v_\alpha^- \tag{26}$$

$$P_\alpha^- v_\alpha^- + \underset{\sigma}{\hat{\epsilon}}_\alpha = \rho_\alpha^- \left(U - v_\alpha^-\right)\left(\epsilon_\alpha^- + \frac{1}{2} (v_\alpha^-)^2\right) + h_\alpha^- \tag{27}$$

Here, we have used U and $\underset{\sigma}{\hat{m}}_\alpha$ to designate the components of $\mathbf{U}$ and $\underset{\sigma}{\hat{\mathbf{m}}}_\alpha$ normal to the shock surface. These three equations are precisely the usual jump relations for a single material except for the supply terms $\underset{\sigma}{\hat{c}}_\alpha$, $\underset{\sigma}{\hat{m}}_\alpha$, and $\underset{\sigma}{\hat{\epsilon}}_\alpha$. These supply terms allow us the versatility to better represent composite materials.

Next we assume each of the constituents has the same particle velocity following the shock. That is

$$v_1^- = v_2^- = \cdots = v_k^- = v^- \tag{28}$$

The last equality is obtained from Equation (4). This assumption implies that the constituents may not diffuse and we have from Equation (5)

$$u_\alpha = 0, \qquad \alpha = 1, 2, \ldots, k \tag{29}$$

When no diffusion occurs, the definitions for total pressure, internal energy, and heat flux are greatly simplified. Using Equations (21) and (29) in Equations (7), (8), and (9), we now have

$$P = \sum_\alpha P_\alpha \tag{30}$$

$$\epsilon = \sum_\alpha c_\alpha \epsilon_\alpha \tag{31}$$

$$h = \sum_\alpha h_\alpha \tag{32}$$

Equation (28) also yields certain other simplifications. Equation (25) can now be written as

$$\eta_\alpha = \frac{U}{U - v^-} + \frac{\hat{c}_\alpha}{\rho_{\alpha_0}(U - v^-)} \tag{33}$$

where η_α is the compression in S_α

$$\eta_\alpha = \rho_\alpha^- / \rho_{\alpha_0} \tag{34}$$

and we have assumed $\rho_{\alpha_0} \neq 0$. Summing Equation (25) over α and using Equation (13), we obtain the following familiar result for the whole mixture.

$$\eta = \frac{U}{U - v^-} \tag{35}$$

where

$$\eta = \rho^- / \rho_0 \tag{36}$$

We conclude, then, that whenever the mass supply, $\underset{\sigma}{\hat{c}}_\alpha$, vanishes, the compression, η_α, is independent of α.

$$\eta_\alpha = \eta, \quad \text{for all } \alpha \text{ such that } \underset{\sigma}{\hat{c}}_\alpha = 0 \tag{37}$$

That is, each constituent is equally compressed. Of course, the compression, η_α, is defined in terms of mixture density, ρ_α, rather than the crystal

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density, $\tilde{\rho}_\alpha$. No relation such as Equation (37) can be obtained for the constituent compression in the physical configuration, $\tilde{\rho}_\alpha^-/\tilde{\rho}_{\alpha_0}$. When $\hat{c}_\alpha$ is not

zero, Equation (37) must be replaced by

$$\eta_\alpha = \eta \left(1 + \frac{\hat{c}_\alpha}{\rho_{\alpha_0} U} \right) \tag{38}$$

We can now investigate the question of a phase transformation in one of the constituents. Suppose constituent S_1 transforms into constituent S_2 at shock pressures in excess of $\tilde{P}^*$. We assume that no S_2 is initially present leading the shock. That is

$$\rho_{2_0} = 0 \tag{39}$$

We assume that, following the shock, all of constituent S_1 has been transformed to S_2 so that

$$\rho_1^- = 0 \tag{40}$$

Using Equation (40) in Equation (25), we obtain

$$\hat{c}_1 = -\rho_{1_0} U, \qquad \text{for} \quad \tilde{P}_\alpha^- \geq \tilde{P}^* \tag{41}$$

while use of Equation (39) in Equation (25) gives

$$\hat{c}_2 = \rho_2^-(U - v^-), \qquad \text{for} \quad \tilde{P}_\alpha^- \geq \tilde{P}^* \tag{42}$$

Then, if no other mass exchange occurs, Equation (13) implies that

$$\rho_2^-(U - v^-) = \rho_{1_0} U \tag{43}$$

whenever $\tilde{P}_\alpha^-$ exceeds the transformation pressure, $\tilde{P}^*$. In this case, we have

$$\frac{\rho_2^-}{\rho_{1_0}} = \eta_3 = \eta_4 = \cdots = \eta_k = \eta \tag{44}$$

More complex interactions, including chemical reactions among constituents within the shock surface, can be treated in a similar manner. It is necessary only to assume constitutive relations for the supplies, $\hat{c}_\alpha$, obeying Equation (13).

Next, we turn to the physical configuration and assume that the constituent crystal pressures, $\tilde{P}_\alpha^-$, are all equal, provided S_α exists behind the shock. That is

$$\tilde{P}_\alpha^- = \tilde{P}^-, \qquad \text{for all } \alpha \text{ such that } \rho_\alpha^- \neq 0 \tag{45}$$

Then Equations (23) and (30), together with this assumption, imply that

$$P^- = \widetilde{P}^- \tag{46}$$

That is, the total mixture pressure must equal the crystal pressure in each constituent. The momentum jump equation [Equation (26)] can then be written as

$$n_\alpha^- \widetilde{P}^- + \underset{\sigma}{m_\alpha} = \rho_\alpha^-(U - v^-)v^- \tag{47}$$

Summing this equation over α and using Equations (14) and (16), we obtain the usual momentum jump equation for the whole mixture.

$$\widetilde{P}^- = \rho^-(U - v^-)v^- \tag{48}$$

Our assumptions of equal particle velocities and equal crystal pressures in all extant constituents have implied the form of the momentum supply functions, $\underset{\sigma}{\hat{m}_\alpha}$. Eliminating the quantity $(U - v^-)v^-$ between Equations (47) and (48) yields

$$\underset{\sigma}{\hat{m}_\alpha} = \widetilde{P}^-(c_\alpha^- - n_\alpha^-) \tag{49}$$

where c_α^- is the concentration ρ_α^-/ρ^- following the shock. Using Equation (38), we can write c_α^- as

$$c_\alpha^- = c_{\alpha 0} \left(1 + \frac{\underset{\sigma}{\hat{c}_\alpha}}{\rho_{\alpha 0} U} \right) \tag{50}$$

Note that, if no mass transfer occurs, the concentration will not change crossing the shock.

The energy balance relation, Equation (27), may also be simplified by these assumptions. If S_α does not vanish behind the shock, we can divide Equation (27) by the volume fraction, n_α^-, to obtain

$$\widetilde{P}^- v^- + \frac{\underset{\sigma}{\hat{\epsilon}_\alpha}}{n_\alpha^-} = \tilde{\rho}_\alpha^-(U - v^-)\left(e_\alpha^- + \frac{1}{2}(v^-)^2 \right) + \tilde{h}_\alpha^- \tag{51}$$

where Equations (17), (19), (23), (28), and (45) have all been used. Also, summing Equation (27) over α and taking Equations (15), (30), (31), and (32) into account, we obtain the usual energy jump equation for the whole mixture.

$$\widetilde{P}^- v^- = \rho^-(U - v^-)\left(\epsilon^- + \frac{1}{2}(v^-)^2 \right) + h^- \tag{52}$$

We point out here that the constituent heat flux, h_α^-, does not account in any way for transfer of heat between constituents. The action of h_α^- is entirely restricted to S_α. Heat may be transferred among the constituents, however,

and this exchange is manifested in the energy supply, $\hat{\epsilon}_\alpha$. In the absence of heat conduction and energy transfer (i.e. $h_\alpha^- = 0$, $\hat{\epsilon}_\sigma = 0$, $\alpha = 1, 2, \ldots, k$), Equations (51) and (52) combine to yield a particularly simple relation between the total energy density for any constituent and that for the whole mixture.

$$c_\alpha^- \left(\epsilon_\alpha^- + \frac{1}{2} (v^-)^2 \right) = n_\alpha^- \left(\epsilon^- + \frac{1}{2} (v^-)^2 \right) \tag{53}$$

Equations (3) and (17), in addition to (51) and (52), have been used to obtain this relation.

CONCLUSIONS

The simplified theory presented in the preceeding section can now be completed by the addition of certain constitutive relations. The resulting set of coupled algebraic equations may then be solved to obtain any of the usual Hugoniot descriptions ($P^- - \eta$, $P^- - v^-$, $U - v^-$, etc.) for the whole mixture. In this manner, the Hugoniot for a composite material may be constructed from known equation-of-state data for its constituents, plus constitutive relations governing mass and energy transfer, and heat conduction.

The simplest possible theory occurs when none of the mixture constituents is heat conducting, and no mass or energy transfer occurs. That is

$$h_\alpha^- = \hat{c}_\alpha = \hat{\epsilon}_\alpha = 0, \qquad \alpha = 1, 2, \ldots, k \tag{54}$$

This case is equivalent to the adiabatic theory of Tsou and Chou [3]. No heat is transferred between constituents, and phase transformations cannot be considered.

The following equations result from the assumptions of Equation (54). Jump relations for mass, momentum, and energy for the whole mixture are

$$\rho^- (U - v^-) = \rho_0 U \tag{55}$$

$$\widetilde{P}^- = \rho^- (U - v^-) v^- \tag{48}$$

$$\widetilde{P}^- v^- = \rho^- (U - v^-) \left(\epsilon^- + \frac{1}{2} (v^-)^2 \right) \tag{56}$$

These three equations are then augmented by the energy balance relation for each of the constituents obtained from Equation (53).

$$c_{\alpha_0} \left(\epsilon_\alpha^- + \frac{1}{2} (v^-)^2 \right) = n_\alpha^- \left(\epsilon^- + \frac{1}{2} (v^-)^2 \right), \qquad \alpha = 1, 2, \ldots, k \tag{57}$$

Note that, when no mass transfer occurs, the concentration, c_α^-, following the shock must equal the initial concentration, c_{α_0} [Equation (50)]. The volume

fraction, n_α^-, can be related to the constituent crystal density, $\tilde{\rho}_\alpha^-$, by combining Equations (3) and (17). Thus

$$n_\alpha^- = \frac{c_{\alpha_0}\rho^-}{\tilde{\rho}_\alpha^-}, \qquad \alpha = 1, 2, \ldots k \tag{58}$$

while Equation (20) relates the mixture and constituent densities

$$\frac{1}{\rho^-} = \sum_\alpha \frac{c_{\alpha_0}}{\tilde{\rho}_\alpha^-} \tag{59}$$

Finally, we have the equation-of-state for each of the constituents.

$$\widetilde{P}^- = F_\alpha(\tilde{\rho}_\beta^-, \epsilon_\beta^-), \qquad \alpha, \beta = 1, 2, \ldots, k \tag{60}$$

Equations (55) through (60) now constitute $3k + 4$ equations in $3k + 5$ unknowns: $\rho^-, U, v^-, \widetilde{P}^-, \epsilon^-, n_\alpha^-, \tilde{\rho}_\alpha^-, \epsilon_\alpha^-, \alpha = 1, 2, \ldots, k$. As in any Hugoniot description, one variable is specified in order to solve for all others.

This specialized theory may now be expanded by eliminating any of the three assumptions of Equation (54). For example, if one wishes to consider heat conduction within the constituents, it is necessary to assume constitutive relations for the fluxes, $\tilde{h}_\alpha^-$, and to replace Equation (57) by

$$\tilde{\rho}_\alpha^-(U - v^-)\left(\epsilon_\alpha^- + \frac{1}{2}(v^-)^2\right) + \tilde{h}_\alpha^- = \rho^-\left(\epsilon^- + \frac{1}{2}(v^-)^2\right) + h^- \tag{61}$$

subject to Equation (32). On the other hand, if energy transfer among constituents is desired, Equation (57) is replaced by

$$c_{\alpha_0}\left(\epsilon_\alpha^- + \frac{1}{2}(v^-)^2\right) = n_\alpha^-\left(\epsilon^- + \frac{1}{2}(v^-)^2\right) + \frac{\hat{\epsilon}_\alpha}{\sigma}{\rho_0 U} \tag{62}$$

Then it is necessary to postulate constitutive equations for the energy supply terms, $\hat{\epsilon}_\alpha$.
σ

$$\hat{\epsilon}_\alpha = E_\alpha(\tilde{\rho}_\beta^-, \epsilon_\beta^-), \qquad \alpha, \beta = 1, 2, \ldots, k \tag{63}$$

subject to Equation (15). Finally, if mass transfer is to be considered, Equation (57) is replaced by Equation (53). Equations (20) and (50) are also employed and constitutive relations of the form

$$\hat{c}_\alpha = C_\alpha(\tilde{\rho}_\beta^-, \epsilon_\beta^-), \qquad \alpha, \beta = 1, 2, \ldots, k \tag{64}$$

can be assumed. Any two, or all three, of the assumptions in Equation (54) may be easily dropped by combining the equations noted above.

In application to a particular composite material, the constitutive relations [Equations (63) and (64)] may be formulated in any manner consistent

with Equations (13) and (15). Equation (64) should be primarily of interest when constituent phase transformations seem likely. The form of Equation (64) applicable to this case has been outlined in the preceding section. Equation (63) can be employed to effect the transfer of thermal energy between constituents. A possible form for Equation (63) is a linear combination of the internal energy differences.

$$\hat{\epsilon}_{\alpha}_{\sigma} = \sum_{\beta} f_{\beta\alpha}(\epsilon_{\bar{\beta}} - \epsilon_{\bar{\beta}}) \tag{65}$$

Equation (15) is then satisfied whenever

$$\sum_{\beta} (f_{\beta\alpha} - f_{\alpha\beta}) = 0 \tag{66}$$

More complex forms for Equation (63) can be postulated employing the constituent temperatures.

In a future paper, we shall employ the theory presented above to represent certain two- and three-constituent mixtures and compare this representation with experimental Hugoniot results.

NOMENCLATURE

N   = unit normal vector to shock surface
P = pressure
T   = stress tensor
U   = velocity vector of shock surface
X_{α} = particle of S_{α}
c_{α} = concentration or mass fraction for S_{α}
$\hat{c}_{\alpha}_{\sigma}$ = mass supply for S_{α}
h   = heat flux
$\mathbf{m}_{\alpha}_{\sigma}$ = momentum supply for S_{α}
n_{α} = volume fraction for S_{α}
S_{α} = constituent α
t = time
$\mathbf{u}_{\alpha}$ = diffusion velocity for S_{α}
v   = velocity vector
x   = position
ϵ = internal energy
$\hat{\epsilon}_{\alpha}_{\sigma}$ = energy supply for S_{α}
η = compression
ρ = mass density
[ ]   = denotes jump crossing shock

Subscripts

α, β = identify constituent
0   = initial value

Superscripts

+   = value leading shock
−   = value following shock

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